國立清華大學數學系碩士班純數組

碩士論文

無限正交群之 K 理論的穩定分解

Stable splitting of the complex connective K-theory of the infinite special orthogonal group

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Abstract

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目錄

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1 Introduction

Let bu be the complex connective K-theory and SO be the infinite special orthogonal group. Then the purpose of this paper is to give the stable splitting of $bu \wedge SO$.

Let $HZ/2$ be the $Z/2$ Eilenberg-Mac Lane spectrum, $BO(n)$ be the classfying space of the n-th orthogonal group, BO be the classfying space of the infinite orthogonal group, and $RP^{\infty} = BO(1)$ be the infinite projective space. Let $H^*(X)$, $\tilde{H}^*(X)$ be the unreduced mod 2 cohomology and the reduced mod 2 cohomology of X respectively, and let $H_*(X)$, $\tilde{H}_*(X)$ be the unreduced mod 2 homology and the reduced mod 2 homology of X respectively. For simplicity of notations, we write $\otimes$ instead of $\otimes_{Z/2}$. Throughout this paper the homotopy equivalences, the spaces, or the spectra are localized at prime 2.

Recall that

$$H^*(BO(n)) = Z/2\,[\omega_1,\,\omega_2,\,\cdots,\,\omega_n]$$

is the polynomial ring generated by $\omega_1, \omega_2, \cdots, \omega_n$, where ω_i is the i-th Stiefel-Whitney class. In particular,

$$H^*(BO) = Z/2\,[\omega_1,\,\omega_2,\,\cdots]$$

and

$$H^*(RP^{\infty}) = H^*(BO(1)) = Z/2\,[\omega_1]\,.$$

Then let $b_i \in H_i(RP^{\infty})$ be the dual class of $\omega_1^i \in H^*(RP^{\infty})$, $i \geq 0$. Hence

$$H_*(BO) = Z/2[b_1,\,b_2,\,b_3,\cdots],$$

where $\deg(b_i) = i$, $i \geq 0$, and $b_i = \eta_*(b_i)$, where $\eta : RP^{\infty} \to BO$ is the natural inclusion map.

Let SU be the infinite special unitary group. Then we have the fibration sequence

$$\Omega\,(SO) \to \Omega\,(SU) \to \Omega\,(SU/SO) \to SO \xrightarrow{\subseteq} SU \to SU/SO.$$

By Bott periodicity, $\Omega\,(SU/SO)$ is homotopic to BO, so we have the map $\phi : BO \to SO$. Recall that

$$H^*(SO) = Z/2\,[e_1,\,e_3,\,e_5,\,\cdots,\,e_{2i-1},\,\cdots]$$

is the polynomial ring generated by $e_1, e_3, e_5, \cdots, e_{2i-1}, \cdots$, where $\deg(e_{2i-1}) = 2i-1$, $i \geq 1$, and

$$H_*(SO) = Z/2\langle b_1, b_2, b_3, \cdots \rangle,$$

is the exterior algebra generated by $b_1, b_2, b_3, \cdots$, where $\deg(b_i) = i$, $i \geq 0$, and $b_i = \phi_* \eta_*(b_i)$.

Also recall that $bu_* = Z_{(2)}[v_1]$, where $\deg(v_1) = 2$, and

$$H^*(bu) \cong A//A(Q_0, Q_1) \cong A \otimes_E Z/2,$$

where A is the mod 2 steenrod algebra, $A(Q_0, Q_1)$ is the ideal of A generated by $Q_0 = Sq^1$, $Q_1 = Sq^3 + Sq^2Sq^1$ and $E = Z/2\langle Q_0, Q_1\rangle$, which is the exterior algebra generated by the Milnor generators, is a subalgebra of A. By the Cartan formula $Sq^i(xy) = \sum_{j=0}^{i} Sq^j(x)Sq^{i-j}(y)$, we know Q_0 and Q_1 act as derivations, that is, $Q_k(xy) = Q_k(x)y + xQ_k(y)$, $k = 0$ or 1. Moreover, since for any space X, $\tilde{H}^*(X)$ is an E-module, we can say an element in $\tilde{H}^*(X)$ is indecomposable or decomposable.

E. Ossa [7] has showed that

$$bu \wedge RP^\infty \wedge RP^\infty \simeq [\bigvee_{0<i,j} \Sigma^{2i+2j-2} HZ/2] \vee [\Sigma^2 bu \wedge RP^\infty].$$

Also, D. Y. Yan [11] splits $bu \wedge BO$ to the suspended copies of $HZ/2$, bu, and $bu \wedge RP^\infty$. Via this spitting, we will get the splitting of $bu \wedge SO$.

For the e_i generator as above, we have the formula [6]

$$Sq^j(e_i) = \binom{i}{j} e_{i+j},$$

where $e_{2n} = e_n^2$ for all positive integer n, $\binom{i}{j}$ is the binomial coefficient. In this paper, we will find an E-linearly independent subset $T_S = \{t_i \mid i \in \Lambda_S\}$ of $\tilde{H}^*(SO)$ such that $\tilde{H}^*(SO) \cong M_S \oplus D_S^*$, where M_S is the free E-submodule of $\tilde{H}^*(SO)$ generated by T_S and $D_S^* \cong \tilde{H}^*(SO)/M_S$ doesn't contain any free generator. Note that T_S is not empty since the element $e_1e_3 \in \tilde{H}^*(SO)$ has

$$\begin{aligned} Q_0(e_1e_3) &= e_1^5 + e_1^2 e_3, \\ Q_1(e_1e_3) &= e_1^4 e_3 + e_1 e_3^2, \\ Q_0Q_1(e_1e_3) &= e_1^8 + e_1^2 e_3^2. \end{aligned}$$

For $t_i \in T_S$ with degree τ_i, let t_i be represented by $g_{t_i}: SO \to \Sigma^{\tau_i} HZ/2$. Let $1 \in H^0(bu)$ be represented by $i: bu \to HZ/2$ which is the multiplicative map and $\mu\prime$ be the ring structure map of $HZ/2$. We have the following composite map:

$$g: bu \wedge SO \overset{bu \wedge \bigvee_{t_i} g_{t_i}}{\to} bu \wedge [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2] \overset{\bigvee_{t_i} \nu}{\to} [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2],$$

where $\nu: bu \wedge HZ/2 \overset{i \wedge HZ/2}{\to} HZ/2 \wedge HZ/2 \overset{\mu\prime}{\to} HZ/2$. Let W be the stable fibre of g, that is, we have the stable cofibration

$$W \overset{h}{\to} bu \wedge SO \overset{g}{\to} [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2].$$

Now we state the main result of this paper.

Theorem A. There is a stable spliting

$$bu \wedge SO \simeq [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2] \vee W,$$

where $\tau_i = \deg t_i$, for each $t_i \in T_S = \{t_i \mid i \in \Lambda_S\}$, that is, one copy $\Sigma^{\tau_i} HZ/2$ corresponds to one element t_i in T_S. As an A-module, there is no any free submodule of $H^*(W)$.

Remark. Every element in $\widetilde{bu_*}SO$, which is both a 2-torsion and a v_1-torsion, is not in $\pi_*(W)$.

To prove Theorem A, the first step is to construct the E-linearly independent subset $T_S = \{t_i \mid i \in \Lambda_S\}$ of $\tilde{H}^*(SO)$ and the composite map

$$f : [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2] \to bu \wedge BO \overset{bu\wedge\phi}{\to} bu \wedge SO,$$

where the first map is induced from the splitting in [11]. Then by the cofibration

$$W \overset{h}{\to} bu \wedge SO \overset{g}{\to} [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2],$$

we have the map

$$F = f \vee h : [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2] \vee W \to bu \wedge SO.$$

Finally we prove the map F induces an isomorphism on the mod 2 cohomologies. Hence F is a homotopy equivalence, and this completes the proof of Theorem A.

The rest of this paper is organized as follows : In section 2, we will construct the subset T_S of $\tilde{H}^*(SO)$ and the stable map $[\bigvee_{t_i} \Sigma^{\tau_i} HZ/2] \to bu \wedge SO$. In section 3, we will prove Theorem A.

2 The E-linearly independent set T_S

In this section we will construct the E-linearly independent subset $T_S = \{t_i \mid i \in \Lambda_S\}$ of $\tilde{H}^*(SO)$ and the stable map

$$f : [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2] \to bu \wedge SO.$$

First we recall what we need. Suppose M and N are left A-modules with actions μ_M and μ_N, then $M \otimes N$ is also a left A-module with the action defined by the composite

$$A \otimes M \otimes N \overset{\psi\otimes M\otimes N}{\to} A \otimes A \otimes M \otimes N \overset{A\otimes T\otimes N}{\to} A \otimes M \otimes A \otimes N \overset{\mu_M\otimes\mu_N}{\to} M \otimes N,$$

where ψ is the diagonal map of A and $T(a \otimes b) = (-1)^{\dim a \dim b}(b \otimes a)$ is the twist map. We write ${}_D(M \otimes N)$ to indicate $M \otimes N$ with this left action. Morever, ${}_L(M \otimes N)$ indicates the extended A action over M.

Proposition 2.1. (Proposition 1.7 of [3]) If B is a Hopf subalgebra of A, M a left A-module, N a left B-module, then

$${}_D[M \otimes (A \otimes_B N)] \cong_L [A \otimes_B {}_D(M \otimes N)]$$

as left A-modules.

Since B is a subalgebra of A, we know that M is a left B-module. Heence ${}_D(M \otimes N)$ is a left B-module with the action:

$$B \otimes M \otimes N \overset{\psi|_B \otimes M \otimes N}{\rightarrow} B \otimes B \otimes M \otimes N \overset{B \otimes T \otimes N}{\rightarrow} B \otimes M \otimes B \otimes N \overset{\mu_M|_B \otimes \mu_N}{\rightarrow} M \otimes N,$$

where $\psi|_B$ is the diagonal map of A restricted on B and $\mu_M|_B$ is the action of M restricted on B. Also we know that A is both a right B-module and a left A-module, hence $A \otimes_B N$ is a left A-module with the extended action over A. For the detail proof we refer the reader to [3].

Note 2.2. Let N be $Z/2$ and B be E in proposition 2.1. Since

$${}_D[M \otimes (A \otimes_E Z/2)] \cong_D [(A \otimes_E Z/2) \otimes M] \; and \; {}_D(M \otimes Z/2) \cong M,$$

this isomorphism (see [1] and Proposition 1.1 of [3])

$$\theta :_L [A \otimes_E M] \overset{\cong}{\rightarrow}_D [(A \otimes_E Z/2) \otimes M]$$

is given by $\theta(a \otimes x) = \Sigma a' \otimes 1 \otimes a'' x$, with the inverse $\theta^{-1}(a \otimes 1 \otimes x) = \Sigma a' \otimes \chi(a'')x$, where $\psi(a) = \Sigma a' \otimes a''$ and χ is the conjugation map.

Since $H^*(bu) \cong A//A(Q_0, Q_1) \cong A \otimes_E Z/2$, from Note 2.2, we have

$$H^*(bu \wedge X) \cong H^*(bu) \otimes \tilde{H}^*(X) \cong (A \otimes_E Z/2) \otimes \tilde{H}^*(X) \overset{\theta^{-1}}{\cong} A \otimes_E \tilde{H}^*(X)$$

for any space X.

Theorem 2.3. There is a E-linearly independent subset $T_S = \{t_i \mid i \in \Lambda_S\}$ of $\tilde{H}^*(SO)$ and a submodule D_S^* of $\tilde{H}^*(SO)$ such that $\tilde{H}^*(SO) \cong M_S \oplus D_S^*$, where M_S is the free E-submodule of $\tilde{H}^*(SO)$ generated by T_S. Moreover, there is the spectrum $[\bigvee_{t_i} \Sigma^{\tau_i} HZ/2]$, one copy $\Sigma^{\tau_i} HZ/2$ corresponding to one $t_i \in T_S$ with degree τ_i, and a stable map

$$f : [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2] \rightarrow bu \wedge SO,$$

such that the following composition

$$\begin{aligned} \vartheta : A \otimes_E M_S &\overset{\subseteq}{\rightarrow} A \otimes_E (M_S \oplus D_S^*) \cong A \otimes_E \tilde{H}^*(SO) \overset{\theta}{\cong} (A \otimes_E Z/2) \otimes \tilde{H}^*(SO) \\ \cong \quad & H^*(bu) \otimes \tilde{H}^*(SO) \cong H^*(bu \wedge SO) \overset{f^*}{\rightarrow} H^*(\bigvee_{t_i} \Sigma^{\tau_i} HZ/2) \end{aligned}$$

is an ismorphism, and $A \otimes_E D_S^*$ has no any free A-submodule.

Since to prove Theorem 2.3, we strongly rely on the stable splitting of $bu \wedge BO$, we first recall the stable splitting of $bu \wedge BO$ in [11].

Let T_B be the set consisting of all the indecomposable elements $d_j \in \tilde{H}^*(BO)(\, j \in \Lambda_B \,)$ such that d_j is the finite sum of monomials, d_j does not contain these two kinds of monomials $\omega_2^{2m_1} \omega_4^{2m_2} \cdots \omega_{2k}^{2m_k}$, $\sum_{i=1}^{k} m_i > 0$, and $\omega_1^j \omega_2^{2m_1} \omega_4^{2m_2} \cdots \omega_{2t}^{2m_t}$, $\sum_{i=1}^{t} m_i \geq 0$, $j \geq 1$, and such that any finite sum of such elements d_j and $\omega_1^{2j-1} \omega_2^{2m_1} \omega_4^{2m_2} \cdots \omega_{2t}^{2m_t}$,

$\sum_{i=1}^{t} m_i \geq 0$, $j \geq 1$, is not decomposable, that is, $(\sum_{finite} d_j)$ or $[(\sum_{finite} d_j) + (\sum_{finite,\ j\geq 1} \omega_1^{2j-1}\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2t}^{2m_t})]$ is not decomposable, that is, we have the Adams E_2-terms (for $\widetilde{bu_*BO}$) $E_2^{0,\,*} = \{[d_j],\ [\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2k}^{2m_k}],\ [\omega_1^{2j-1}\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2t}^{2m_t}] \mid j \in \Lambda_n,\ \sum_{i=1}^{k} m_i > 0,\ \sum_{i=1}^{t} m_i \geq 0,\ j \geq 1\}$.

Theorem 2.4. (Theorem 3.3 in [11]) As an E-module, $\tilde{H}^*(BO)$ is isomorphic to $D_1^* \oplus D_2^* \oplus M_B$, where D_1^* is an E-module with E-generators $\{\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2t}^{2m_t} \mid \sum_{i=1}^{k} m_i > 0\}$, D_2^* is an E-module with E-generators $\{\omega_1^{2j-1}\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2t}^{2m_t} \mid \sum_{i=1}^{t} m_i \geq 0,\ j \geq 1\}$, and M_B is isomorphic to the free E-module $\tilde{H}^*(BO)/(D_1^* \oplus D_2^*)$ with E-basis $T_B = \{d_j \mid j \in \Lambda_B\}$.

Thus we have $H^*(bu \wedge BO) \cong A \otimes_E \tilde{H}^*(BO) \cong A \otimes_E (D_1^* \oplus D_2^* \oplus M_B)$.

Note. We have

$$\begin{aligned} Q_0Q_1(\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2k}^{2m_k}) &= 0, \\ Q_0Q_1(\omega_1^{j}\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2t}^{2m_t}) &= Q_0Q_1(\omega_1^{j})\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2t}^{2m_t} = 0, \end{aligned}$$

that is, for each $x \in \tilde{H}^*(BO)$, $Q_0Q_1(x) \neq 0$ if and only if as a sum of monomials, x contains some elements in T_B.

Theorem 2.5. (Theorem A in [11]) There is a stable splitting

$$bu \wedge BO \simeq [\bigvee_{d_j} \Sigma^{\alpha_j} HZ/2] \vee [\bigvee_{\beta} \Sigma^{\beta} bu] \vee [\bigvee_{\gamma} \Sigma^{\gamma} bu \wedge RP^{\infty}],$$

where $\alpha_j = \deg d_j$, for each indecomposable elements $d_j \in T_B = \{d_j \mid j \in \Lambda_B\} \subseteq \tilde{H}^*(BO)$, that is, one copy $\Sigma^{\alpha_j} HZ/2$ corresponds to one indecomposable elements d_j, $j \in \Lambda_B$, $\beta = \sum_{i=1}^{k} 4im_i$, $\sum_{i=1}^{k} m_i > 0$ for the nonnegative integers m_i, that is, one copy $\Sigma^{\beta} bu$ corresponds to one monomial $\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2k}^{2m_k}$, $\sum_{i=1}^{k} m_i > 0$ for the nonnegative integers m_i, and $\gamma = \sum_{i=1}^{t} 4im_i$, $\sum_{i=1}^{t} m_i \geq 0$ for the nonnegative integers m_i, that is, one copy $\Sigma^{\gamma} bu \wedge RP^{\infty}$ corresponds to one monomial $\omega_2^{2m_1}\omega_4^{2m_2}\cdots\omega_{2k}^{2m_k}$, $\sum_{i=1}^{t} m_i \geq 0$.

By Theorem 2.5, we have the homotopy equivalence

$$\Phi : [\bigvee_{d_j} \Sigma^{\alpha_j} HZ/2] \vee [\bigvee_{\beta} \Sigma^{\beta} bu] \vee [\bigvee_{\gamma} \Sigma^{\gamma} bu \wedge RP^{\infty}] \to bu \wedge BO$$

such that $\Phi^*(1 \otimes d_j) = \Sigma^{\alpha_j} 1_j \in H^*(\bigvee_{d_j} \Sigma^{\alpha_j} HZ/2)$ for each $d_j \in T_B = \{d_j \mid j \in \Lambda_B\}$ with degree α_j. Let φ be the restriction of Φ on $[\bigvee_{d_j} \Sigma^{\alpha_j} HZ/2]$, then φ^* takes the free A-module $A \otimes_E M_B$ isomorphically onto $H^*(\bigvee_{d_j} \Sigma^{\alpha_j} HZ/2)$.

Lemma 2.6. For each dimension $m \geq 1$, there is a subset $T_m = \{t_{m,i} \mid 1 \leq i \leq l_m\}$ of $\tilde{H}^m(SO)$, a spectrum $[\bigvee_{t_{m,i} \in T_m} \Sigma^{\tau_i} HZ/2]$, each copy $\Sigma^{\tau_i} HZ/2$ corresponding to a $t_{m,i} \in T_m$, where $\tau_i = m$, and a stable map

$$f_m : [\bigvee_{t_{m,i} \in T_m} \Sigma^{\tau_i} HZ/2] \to bu \wedge SO,$$

such that we have

$$f_m^*(1 \otimes t_{m,i}) = \Sigma^{\tau_i} 1,$$

where $t_{m,i} \in T_m$ and $\Sigma^{\tau_i} 1 \in H^m(\bigvee_{t_{m,i} \in T_m} \Sigma^{\tau_i} HZ/2)$. Morever, let v_m be the homomorphism defined as the following composition

$$H^*(bu \wedge SO) \overset{(bu\wedge\phi)^*}{\longrightarrow} H^*(bu \wedge BO) \overset{\varphi^*}{\longrightarrow} H^*(\bigvee_{d_j} \Sigma^{\alpha_j} HZ/2) \overset{p}{\longrightarrow} H^*(\bigvee_{d_j \in T_{B,m}} \Sigma^{\alpha_j} HZ/2),$$

where $T_{B,m} = \{d_j \in T_B \mid j \in \Lambda_B, \deg d_j = m\}$ and p is the projection, then for each element $c \in \tilde{H}^m(SO)$, $v_m(1 \otimes c)$ can be generated by $\{v_m(1 \otimes t_{m,i}) \mid t_{m,i} \in T_m\}$ over $Z/2$.

Proof. For each dimension $m \geq 1$, $\tilde{H}^m(SO)$ and $\tilde{H}^m(BO)$ are both finite $Z/2$-module. Let $\tilde{H}^m(SO) = \{c_{m,i} \mid 1 \leq i \leq s_m\}$ and $T_{B,m} = \{d_{m,j} \in T_B \mid \deg d_{m,j} = m, 1 \leq j \leq r_m\}$. Then consider the map $\phi : BO \to SO$, we have

$$\phi_* : H_*(BO) = Z/2[b_1, b_2, b_3, \cdots] \to Z/2\langle b_1, b_2, b_3, \cdots\rangle = H_*(SO)$$

takes b_i to b_i, $i \geq 0$, that is, ϕ_* is surjective, and hence $\phi^* : H^*(SO) \to H^*(BO)$ is injective. For any $c_{m,\,i} \in \tilde{H}^m(SO)$, we have

$$\phi^*(c_{m,i}) = \sum_{j=1}^{r_m} \delta_{i,j} d_{m,j} + x_i + y_i,$$

where $d_{m,j} \in T_{B,m}$, $\delta_{i,j} = 0$ or 1, $1 \leq j \leq r_m$, $y_i \in D_1^* \oplus D_2^*$, and x_i is in the form $\sum Q_0(d_{j'}) + \sum Q_1(d_{j''}) + \sum Q_0 Q_1(d_{j'''})$, for some $d_{j'}, d_{j''}, d_{j'''} \in T_B$. Then following the composite map

$$v_m : H^*(bu \wedge SO) \overset{(bu\wedge\phi)^*}{\longrightarrow} H^*(bu \wedge BO) \overset{\varphi^*}{\longrightarrow} H^*(\bigvee_{d_j} \Sigma^{\alpha_j} HZ/2) \overset{p}{\longrightarrow} H^*(\bigvee_{d_{m,j} \in T_{B,m}} \Sigma^{\alpha_{m,j}} HZ/2),$$

we have the diagram

$$v_m : 1 \otimes c_{m,i} \overset{(bu\wedge\phi)^*}{\longrightarrow} 1 \otimes (\sum_{j=1}^{r_m} \delta_{i,j} d_{m,j} + x_i + y_i) \overset{\varphi^*}{\longrightarrow} \sum_{j=1}^{r_m} (\delta_{i,j} \Sigma^{\alpha_{m,j}} 1 + Q_{i,j}) \overset{p}{\longrightarrow} \sum_{j=1}^{r_m} (\delta_{i,j} \Sigma^{\alpha_{m,j}} 1),$$

where $\Sigma^{\alpha_{m,j}} 1$ corresponds to $d_{m,j}$ and $Q_{i,j}$ is a linearly combination of $\Sigma^{(m-1)} Q_0$, $\Sigma^{(m-3)} Q_1$, $\Sigma^{(m-4)} Q_0 Q_1$, for $1 \leq j \leq r_m$. Consider the matrix-like diagram

$$\left[\begin{array}{cccc|c} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,r_m} & c_{m,1} \\ \delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,r_m} & c_{m,2} \\ \vdots & \vdots & & \vdots & \vdots \\ \delta_{s_m,1} & \delta_{s_m,2} & \cdots & \delta_{s_m,r_m} & c_{m,s_m} \end{array}\right],$$

we can see that the left part of the i-th row is the coordinate vector of $v_m(1 \otimes c_{m,i})$ relative to $\{\Sigma^{\alpha_{m,j}} 1 \mid 1 \leq j \leq r_m\}$, which is the A-basis of $H^*(\bigvee_{d_{m,j} \in T_{B,m}} \Sigma^{\alpha_{m,j}} HZ/2)$. Assume that there are at most l_m linearly independent rows in the left part, then we can use elemantary row operations, with $Z/2$ coeffiecient, to change it into the following diagram

$$\left[\begin{array}{ccccccccccccc|c}
0 & \cdots & 0 & 1_{1,j_1} & \delta'_{1,j_1+1} & \cdots & 0 & \delta'_{1,j_2+1} & \cdots & 0 & \delta'_{1,j_{l_m}+1} & \cdots & \delta'_{1,r_m} & t_{m,1} \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & 1_{2,j_2} & \delta'_{2,j_2+1} & \cdots & 0 & \delta'_{2,j_{l_m}+1} & \cdots & \delta'_{2,r_m} & t_{m,2} \\
 & & & & & & & & \vdots & & & & & \vdots \\
0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 1_{l_m,j_{l_m}} & \delta'_{l_m,j_{l_m}+1} & \cdots & \delta'_{l_m,r_m} & t_{m,l_m} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & t_{m,l_m+1} \\
\vdots & & & & & & & & & & & & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & t_{m,s_m}
\end{array}\right]$$

where j_i is the number such that $1_{i,j_i}$ is the leftest nonzero element in the i-th row and the only nonzero element in the j_i-th column, $1 \leq j_1 < j_2 < \cdots < j_{l_m} \leq r_m$, $\delta'_{i,j} = 0$ or 1, $t_{m,i}$ is a linearly combination of all $c_{m,i} \in \tilde{H}^m(SO)$, $1 \leq i \leq s_m$, that is,

$$v_m(1 \otimes t_{m,i}) = \begin{cases} \Sigma^{\alpha_{m,j_i}} 1 + \sum\limits_{j > j_i \text{ and } j \neq j_k \text{ for } i \leq k \leq l_m} \delta'_{i,j}(\Sigma^{\alpha_{m,j}} 1) & , \ 1 \leq i \leq l_m \\ 0 & , \ l_m + 1 \leq i \leq s_m. \end{cases}$$

Therefore, for any $c_{m,i} \in \tilde{H}^m(SO)$, $v_m(1 \otimes c_{m,i})$ can be generated by $\{v_m(1 \otimes t_{m,1}), v_m(1 \otimes t_{m,2}), \cdots, v_m(1 \otimes t_{m,l_m})\}$.

Let $T_m = \{t_{m,i} \mid 1 \leq i \leq l_m\}$, then we define the map f_m as the following composition

$$f_m : [\bigvee_{t_{m,i} \in T_m} \Sigma^{\alpha_{m,j_i}} HZ/2] \xrightarrow{\eta} [\bigvee_{d_{m,j} \in T_{B,m}} \Sigma^{\alpha_{m,j}} HZ/2] \xrightarrow{\eta'} [\bigvee_{d_j \in T_B} \Sigma^{\alpha_j} HZ/2] \xrightarrow{\varphi} bu \wedge BO \xrightarrow{bu \wedge \phi} bu \wedge SO,$$

where η and η' are the inclusions, $(\eta')^* = p$. Then for $i = 1, 2, \cdots, l_m$, we have $f_m^*(1 \otimes t_{m,i}) = \eta^*(v_m(1 \otimes t_{m,i}) = \Sigma^{\alpha_{m,j_i}} 1$. This completes the proof. ■

Theorem 2.7. The subset $T_S = \bigcup\limits_{m=1}^{\infty} T_m$ of $\tilde{H}^*(SO)$ is E-linearly independent. Let M_S be the free E-submodule of $\tilde{H}^*(SO)$ generated by T_S. There is an E-submodule D_S^* of $\tilde{H}^*(SO)$ such that $\tilde{H}^*(SO) \cong M_S \oplus D_S^*$ and D_S^* contains no any free E-generator.

Proof. Since the set $\{\Sigma^{\alpha_{m,j_i}} 1 \mid 1 \leq i \leq l_m\}$ is an A-basis of $H^*(\bigvee_{t_{m,i} \in T_m} \Sigma^{\alpha_{m,j_i}} HZ/2)$ and $f_m^*(1 \otimes t_{m,i}) = \Sigma^{\alpha_{m,j_i}} 1$ for each $t_{m,i} \in T_m$, the set $\{1 \otimes t_{m,i} \mid t_{m,i} \in T_m\} \subseteq H^*(bu \wedge SO)$ is an A-linearly independent subset of $H^*(bu \wedge SO)$, and hence T_m is an E-linearly independent subset of $\tilde{H}^*(SO)$. If we have the equation

$$\check{\mathrm{T}}_1 + Q_0(\check{\mathrm{T}}_2) + Q_1(\check{\mathrm{T}}_3) + Q_0 Q_1(\check{\mathrm{T}}_4) = 0,$$

where $\check{\mathrm{T}}_1$, $\check{\mathrm{T}}_2$, $\check{\mathrm{T}}_3$, $\check{\mathrm{T}}_4$, are sums of some $t_{m,i} \in T_m$, $t_{m-1,i} \in T_{m-1}$, $t_{m-3,i} \in T_{m-3}$, $t_{m-4,i} \in T_{m-4}$ respectively, then we also have following equations

$$\begin{aligned} Q_0(\check{\mathrm{T}}_1) + Q_0 Q_1(\check{\mathrm{T}}_3) &= 0, \\ Q_1(\check{\mathrm{T}}_1) + Q_1 Q_0(\check{\mathrm{T}}_2) &= 0, \\ Q_0 Q_1(\check{\mathrm{T}}_1) &= 0, \end{aligned}$$

which implies that $\check{\mathrm{T}}_i = 0$ for each i. Therefore, the subset $T_S = \bigcup\limits_{m=1}^{\infty} T_m$ of $\tilde{H}^*(SO)$ is also E-linearly independent.

From the construction of each T_m in Lemma 2.6, we have one $t_{m,i} \in T_S$ which corresponds to one indecomposable element $d_{m,j_i} \in T_B$. Hence there is a free submodule M' of M_B which is isomorphic to M_S. Thus we have the following short exact sequence

$$0 \to D_S^* \xrightarrow{\subseteq} \tilde{H}^*(SO) \xrightarrow{\vartheta} M_S \to 0,$$

where ϑ is the following combination

$$\tilde{H}^*(SO) \xrightarrow{\phi^*} \tilde{H}^*(BO) \xrightarrow{p} M_B \xrightarrow{p'} M' \cong M_S,$$

p and $p\prime$ are projections, and D_S^* is the kernel of ϑ. Since M_S is free, the short exact sequence splits, hence we have

$$\tilde{H}^*(SO) \cong M_S \oplus D_S^*.$$

Assume that D_S^* contains a free E-generator w with degree m, then the subset $\{w,\, t_{m,1},\, t_{m,2},\, \cdots,\, t_{m,l_m}\}$ of $\tilde{H}^*(SO)$ is E-linearly independent, and the subset $\{1\otimes w,\, 1\otimes t_{m,1},\, 1\otimes t_{m,2},\, \cdots,\, 1\otimes t_{m,l_m}\}$ of $H^*(bu\wedge BO)$ is A-linearly independent. For any $c_{m,i} \in \tilde{H}^m(SO)$, the formula

$$\phi^*(c_{m,i}) = \sum_{j=1}^{r_m} \delta_{i,j} d_{m,j} + x_i + y_i$$

in Lemma 2.6 and Note after Theorem 2.4 give that

$$Q_0Q_1\phi^*(c_{m,i}) = Q_0Q_1\left(\sum_{j=1}^{r_m} \delta_{i,j} d_{m,j} + x + y\right) = Q_0Q_1\left(\sum_{j=1}^{r_m} \delta_{i,j} d_{m,j}\right).$$

Hence we have

$$\begin{aligned} & \Phi^*(bu\wedge\phi)^*(1\otimes Q_0Q_1(c_{m,i})) = \sum_{j=1}^{r_m} (\delta_{i,j}\Sigma^{\alpha_{m,j}} Q_0Q_1) \\ = \quad & \eta''\left(\sum_{j=1}^{r_m} (\delta_{i,j}\Sigma^{\alpha_{m,j}} Q_0Q_1)\right) = \eta'' Q_0Q_1 \sum_{j=1}^{r_m} (\delta_{i,j}\Sigma^{\alpha_{m,j}} 1) = \eta'' Q_0Q_1 v_m(1\otimes c_{m,i}), \end{aligned}$$

where η'' is the inclusion from $H^*(\bigvee_{d_{m,j}\in T_{B,m}} \Sigma^{\alpha_{m,j}} HZ/2)$ to $H^*([\bigvee_{d_j} \Sigma^{\alpha_j} HZ/2] \vee [\bigvee_{\beta} \Sigma^{\beta} bu] \vee [\bigvee_{\gamma} \Sigma^{\gamma} bu \wedge RP^\infty])$. Since ϕ^* is injective and Φ^* is an isomorphism, under the map $\Phi^*(bu\wedge\phi)^*$, the image of $\{1\otimes w,\, 1\otimes t_{m,1},\, 1\otimes t_{m,2},\, \cdots,\, 1\otimes t_{m,l_m}\}$ is also A-linearly independent. However, by Lemma 2.6, $v_m(1\otimes w)$ can be generated by $\{v_m(1\otimes t_{m,1}),\, v_m(1\otimes t_{m,2}), \cdots,\, v_m(1\otimes t_{m,l_m})\}$. This leads to a contradiction. ■

Proof of Theorem 2.3. Since D_S^* contains no any free E-generator, by Theorem 2.7, $A\otimes_E D_S^*$ has no free A-submodule.

Now we construct the map f as the following composition

$$f : [\bigvee_{t_{m,i}\in T_S} \Sigma^{\alpha_{m,j_i}} HZ/2] = \bigvee_{m=1}^{\infty} [\bigvee_{t_{m,i}\in T_m} \Sigma^{\alpha_{m,j_i}} HZ/2] \xrightarrow{\vee f_m} \bigvee_{m=1}^{\infty} [bu\wedge SO] \xrightarrow{F} bu\wedge SO,$$

where F is the folding map. Then for any $m \geq 1$, $1 \leq i \leq l_m$, we have $f^*(1\otimes t_{m,i}) = \Sigma^{\alpha_{m,j_i}} 1 + z_{m,i}$, where $z_{m,i}$ is a sum in the form $\sum \Sigma^{(m-1)} Q_0 + \sum \Sigma^{(m-3)} Q_1 + \sum \Sigma^{(m-4)} Q_0Q_1$. Note that the set $\{\Sigma^{\alpha_{m,j_i}} 1 + z_{m,i} \mid m \geq 1, 1 \leq i \leq l_m\}$ is still an A-basis of $H^*(\bigvee_{t_{m,i}\in T_S} \Sigma^{\alpha_{m,j_i}} HZ/2)$.

From Note 2.2, we have the homomorphism

$$\vartheta : A \otimes_E M_S \overset{\subseteq}{\to} A \otimes_E (M_S \oplus D_S^*) \cong A \otimes_E \tilde{H}^*(SO) \overset{\theta}{\cong} (A \otimes_E Z/2) \otimes \tilde{H}^*(SO)$$
$$\cong \quad H^*(bu) \otimes \tilde{H}^*(SO) \cong H^*(bu \wedge SO) \xrightarrow{f^*} H^*(\bigvee_{t_i} \Sigma^{\tau_i} HZ/2).$$

Since $A \otimes_E M_S$ is a free A-module generated by $\{1 \otimes t_{m,i} \mid t_{m,i} \in T_S\}$ and f^* takes $1 \otimes t_{m,i}$ to the generator $\Sigma^{\alpha_{m,j_i}} 1 + z_{m,i}$ of $H^*(\bigvee_{t_{m,i} \in T_S} \Sigma^{\alpha_{m,j_i}} HZ/2)$, these two free A-module have the same rank, and hence ϑ is an ismorphism. This completes the proof. ■

3 Proof of Theorem A

In this section we are going to prove Theorem A.

For simplicity of the subscripts, let $T_S = \{t_i \mid i \in \Lambda_S\}$ and $\Sigma^{\tau_i} HZ/2$ correspond to t_i, where $\tau_i = \deg t_i$. For $t_i \in T_S = \{t_i \mid i \in \Lambda_S\}$ with degree τ_i, let t_i be represented by $g_{t_i} : SO \to \Sigma^{\tau_i} HZ/2$. We have the following composite map:

$$g : bu \wedge SO \xrightarrow{bu \wedge \bigvee_{t_i} g_{t_i}} bu \wedge [\bigvee_{t_i} \Sigma^{\tau_i} HZ/2] \xrightarrow{\bigvee_{t_i} \nu} \bigvee_{t_i} \Sigma^{\tau_i} HZ/2,$$

where $\nu : bu \wedge HZ/2 \xrightarrow{i \wedge HZ/2} HZ/2 \wedge HZ/2 \xrightarrow{\mu\prime} HZ/2$, $i : bu \to HZ/2$ is the multiplicative map, and $\mu\prime$ is the ring structure map of $HZ/2$.

Proof of Theorem A :

Consider the composite map

$$\kappa : H^*(\bigvee_{t_i} \Sigma^{\tau_i} HZ/2) \xrightarrow{g^*} H^*(bu \wedge SO) \cong A \otimes_E (M_S \oplus D_S^*) \xrightarrow{p} A \otimes_E M_S,$$

where p is the projection map. By the construction of g, g^* sends the generator $\Sigma^{\tau_i} 1 \in H^*(\bigvee_{t_i} \Sigma^{\tau_i} HZ/2)$ to $1 \otimes t_i \in H^*(bu \wedge SO)$, for each $t_i \in T_S$. Since $\chi(1) = 1$, if we follow the composite map κ, then we have the following composite map

$$\Sigma^{\alpha} 1 \xrightarrow{g^*} 1 \otimes t_i \to 1 \otimes (t_i \oplus 0) \xrightarrow{p} 1 \otimes t_i.$$

Hence $\kappa(\Sigma^{\tau_i} 1) = 1 \otimes t_i$ for each generator $\Sigma^{\tau_i} 1$ of the free A-module $H^*(\bigvee_{t_i} \Sigma^{\tau_i} HZ/2)$. Since $A \otimes_E M_S$ is also a free A-module with basis $\{1 \otimes t_i \mid i \in \Lambda_S\}$ and has the same rank with $H^*(\bigvee_{t_i} \Sigma^{\tau_i} HZ/2)$, it follows that κ is an isomorphism.

Recall that W is the stable fibre of g, so we have the stable fibration

$$W \xrightarrow{h} bu \wedge SO \xrightarrow{g} \bigvee_{t_i} \Sigma^{\tau_i} HZ/2,$$

and the long exact sequence

$$\cdots \to H^n(\bigvee_{t_i} \Sigma^{\tau_i} HZ/2) \xrightarrow{g^*} H^n(bu \wedge SO) \xrightarrow{h^*} H^n(W) \xrightarrow{\delta} H^{n+1}(\bigvee_{t_i} \Sigma^{\tau_i} HZ/2) \to \cdots.$$

Since the map g^* is injective, the long exact sequence can be replaced by the short exact sequence

$$0 \to H^*(\underset{t_i}{\vee}\Sigma^{\tau_i} HZ/2) \xrightarrow{g^*} H^*(bu \wedge SO) \xrightarrow{h^*} H^*(W) \to 0.$$

Consider the map $f : \underset{t_i}{\vee}\Sigma^{\tau_i} HZ/2 \to bu \wedge SO$ and the homomorphism ϑ defined in Theorem 2.3, since ϑ takes $A \otimes_E M_S$ isomorphically onto $H^*(\underset{t_i}{\vee}\Sigma^{\tau_i} HZ/2)$, we have $f^* g^* = \kappa\vartheta$ is an isomorphism, that is, the short exact sequence is split. Hence $H^*(W) \cong A \otimes_E D_S^*$ has no free A-submodule by Theorem 2.3. Moreover, the map F defined by the following composition:

$$F : [\underset{t_i}{\vee}\Sigma^{\tau_i} HZ/2] \vee W \xrightarrow{f \vee h} (bu \wedge SO) \vee (bu \wedge SO) \xrightarrow{\psi} bu \wedge SO,$$

where ψ is the foldding map, induces the isomorphism of cohomologies from $H^*(bu \wedge SO)$ to $H^*([\underset{t_i}{\vee}\Sigma^{\tau_i} HZ/2] \vee W)$. This completes the proof. $\blacksquare$

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